In this activity you will use vectors to solve a range of problems.
The problems are based on real-life situations and involve forces or motion in more than one dimension.

Information sheet

Scalar quantities have magnitude but no direction.
For example, mass, distance, speed, and temperature are scalar quantities.

Vectors have magnitude and direction. Examples of vectors include displacement, velocity, acceleration, force and momentum.

Components of a vector

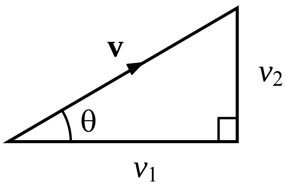
Vectors can be given in terms of perpendicular components.
A two-dimensional vector can be given in terms of two perpendicular unit vectors, **i** and **j**.

For example, suppose the velocity of a yacht has an easterly component of 12 ms^{-1}and a northerly component of 5 ms^{-1}. If **i** represents a unit vector of 1 ms^{-1} to the east and **j** represents a unit vector of 1 ms^{-1} to the north, then the velocity of the yacht can be written as **v** ms^{-1} where **v** = 12**i** + 5**j**.

You can also write this using column vector notation as $\mathbf{v} = \begin{bmatrix} 12 \\ 5 \end{bmatrix}$.

Magnitude and direction of a vector

The diagram shows a vector, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$



Think about

How can you use the triangle to find the magnitude and direction of **v**?

The magnitude of the vector is $v = \sqrt{{v_1}^2 + {v_2}^2}$ and its direction is given by

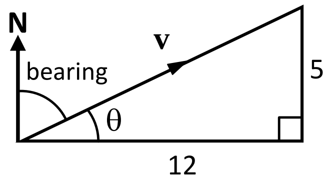
$\theta = \tan^{-1}\left(\dfrac{v_2}{v_1}\right)$.

Note that we often use the same letter for a vector and its magnitude, but the vector is written either in bold or is underlined.

Speed is the magnitude of the velocity vector, so in the case of the yacht, the speed is given by $v = \sqrt{12^2 + 5^2} = \sqrt{169} = 13$ (ms^{-1}).

The direction is given by $\tan\theta = \frac{5}{12} = 0.41\dot{6}$

$\theta = 22.6°$,

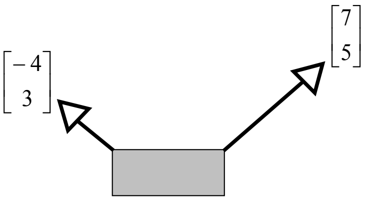


so the velocity vector $\mathbf{v} = \begin{bmatrix} 12 \\ 5 \end{bmatrix}$ represents a speed of 13 ms^{-1} at 22.6° to the north of east.

To give the direction of the yacht's velocity as a bearing: 90° – 22.6° = 67.4°, so the bearing on which the yacht is sailing is 067° (to the nearest degree).

To add or subtract vectors, add or subtract the components.

For example, suppose forces $\mathbf{F_1} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$ and $\mathbf{F_2} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ act on an object as shown.



In this case $\mathbf{i}$ is a horizontal unit vector to the right and $\mathbf{j}$ is a vertical unit vector upwards. Each force is given in newtons (N).

The total force acting on the object is $\begin{bmatrix} 7 \\ 5 \end{bmatrix} + \begin{bmatrix} -4 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$ (N)

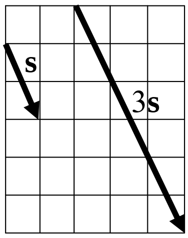
Adding the forces together gives the resultant force acting on the object.

To multiply a vector by a scalar, multiply each component by the scalar.

For example, suppose a displacement $\mathbf{s}$ metres is given by $\mathbf{s} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

The displacement that is three times as far in the same direction is

$3\mathbf{s} = \begin{bmatrix} 3 \\ -6 \end{bmatrix}$.



Think about

What can you say about the components representing $\mathbf{s}$ and $3\mathbf{s}$?

What about any other scalar multiple of $\mathbf{s}$?

Using vectors to solve problems

Problems involving forces and motion in more than one dimension can be solved using vectors. The most useful laws and equations are given below in vector form.

Constant acceleration equations

$\mathbf{v} = \mathbf{u} + \mathbf{a}t$ $\qquad$ $\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$ $\qquad$ $\mathbf{s} = \frac{1}{2}(\mathbf{u} + \mathbf{v})t$

where $\mathbf{u}$ is the initial velocity, $\mathbf{v}$ is the final velocity, $\mathbf{a}$ is the acceleration, t is the time taken and $\mathbf{s}$ is the displacement.

These equations can be used to solve problems such as finding the time when a particle is at a specified position or has a specified velocity. They can also be used to find the position, velocity or acceleration of a particle at a specified time.

Think about

What can you say about the direction of $\mathbf{a}$, $\mathbf{a}t$, and $\mathbf{a}t^2$?
What about $\mathbf{u}$ and $\mathbf{u}t$?

Momentum

Momentum, like velocity, is a vector having both magnitude and direction. When an object of mass m kilograms moves with velocity $\mathbf{v}$ metres per second, its momentum is $m\mathbf{v}$ (where each component is in kg ms^{-1} or Ns).

Forces and Acceleration

Resultant force

When a number of forces act on an object, the resultant force is the sum of these forces. For example, if forces $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ act on an object, then the resultant force is $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$.

Newton's First Law of Motion

A particle will remain at rest or continue to move uniformly in a straight line unless acted upon by a non-zero resultant force.

Newton's Second Law of Motion

A resultant force, $\mathbf{F}$ newtons, acting on a particle of mass m kilograms will give rise to an acceleration, $\mathbf{a}$ ms^{-2}, where $\mathbf{F} = m\mathbf{a}$.

Newton's Third Law of Motion

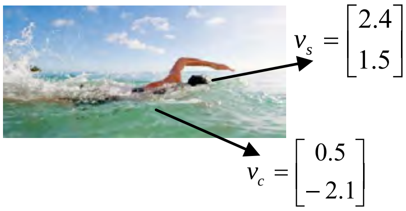
Action and reaction are equal and opposite. This means that if a body A exerts a force on a body B, then B exerts an equal and opposite force on A.

Examples of using vectors to solve problems

Swimmer

Take $\mathbf{i}$ and $\mathbf{j}$ to be unit vectors directed to the east and north respectively.

A woman is swimming with velocity $\mathbf{v_S}$ ms^{-1} where $\mathbf{v_S} = \begin{bmatrix} 2.4 \\ 1.5 \end{bmatrix}$.

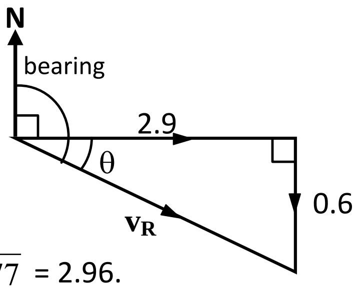


She meets a current with velocity $\mathbf{v_C}$ ms^{-1} where $\mathbf{v_C} = \begin{bmatrix} 0.5 \\ -2.1 \end{bmatrix}$.

Find the magnitude and direction of the swimmer's resultant velocity.
Give the direction as a bearing.

Swimmer: solution

The resultant velocity, $\mathbf{v_R}$, is the sum of the velocity of the woman and the velocity of the current



$$\mathbf{v_R} = \begin{bmatrix} 2.4 \\ 1.5 \end{bmatrix} + \begin{bmatrix} 0.5 \\ -2.1 \end{bmatrix} = \begin{bmatrix} 2.9 \\ -0.6 \end{bmatrix}.$$

The woman's speed (magnitude of $\mathbf{v_R}$) is $v_R = \sqrt{2.9^2 + 0.6^2} = \sqrt{8.77} = 2.96$.

The direction is given by $\theta = \tan^{-1}\left(\frac{0.6}{2.9}\right) = \tan^{-1} 0.2068... = 11.7°$.

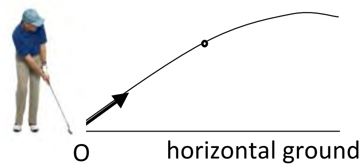
$90°+11.7°= 101.7°$, so the swimmer will travel at 2.96 ms^{-1} on bearing 102° (nearest degree).

Golf ball

Take the unit vectors $\mathbf{i}$ and $\mathbf{j}$ to be horizontal and vertical respectively.

A golf ball is hit from a point O with velocity $\mathbf{u}$ ms^{-1} where $\mathbf{u} = \begin{bmatrix} 25 \\ 16 \end{bmatrix}$.

The golf ball's acceleration is $\mathbf{a}$ ms^{-2} where $\mathbf{a} = \begin{bmatrix} 0 \\ -9.8 \end{bmatrix}$.



Find ...

a an expression for the velocity of the golf ball at time t
b the velocity when $t = 2$.
c the displacement of the ball from O when $t = 2$.

Golf ball: solution

a Using $\mathbf{v}=\mathbf{u}+\mathbf{a}t$ gives $\mathbf{v} = \begin{bmatrix}25\\16\end{bmatrix} + \begin{bmatrix}0\\-9.8\end{bmatrix}t = \begin{bmatrix}25\\16-9.8t\end{bmatrix}$ (ms^{-1}).

b When t = 2, the velocity is $\mathbf{v} = \begin{bmatrix}25\\16-9.8\times 2\end{bmatrix} = \begin{bmatrix}25\\-3.6\end{bmatrix}$ (ms^{-1}).

c Using $\mathbf{s}=\mathbf{u}t+\frac{1}{2}\mathbf{a}t^2$ gives $\mathbf{s} = \begin{bmatrix}25\\16\end{bmatrix}t + \frac{1}{2}\begin{bmatrix}0\\-9.8\end{bmatrix}t^2 = \begin{bmatrix}25t\\16t-4.9t^2\end{bmatrix}$

When t = 2, the displacement from O is $\mathbf{s} = \begin{bmatrix}25\times 2\\16\times 2-4.9\times 2^2\end{bmatrix} = \begin{bmatrix}50\\12.4\end{bmatrix}$ (m).

Skier

Take the unit vectors **i** and **j** to be directed to the east and north respectively.

A skier of mass 60 kg starts skiing across horizontal ground with velocity **u** ms^{-1} where $\mathbf{u} = \begin{bmatrix}-8\\6\end{bmatrix}$.

A constant horizontal force $\mathbf{F} = \begin{bmatrix}24\\-15\end{bmatrix}$ newtons then acts on the skier.

a Find the skier's acceleration.

b Find the speed and direction of the skier 20 seconds later.

Skier solution

a Using **F** = m**a** gives $\begin{bmatrix}24\\-15\end{bmatrix} = 60\mathbf{a}$

Dividing both sides of this equation by 60 gives $\mathbf{a} = \begin{bmatrix}0.4\\-0.25\end{bmatrix}$ (ms^{-2}).

b Using $\mathbf{v}=\mathbf{u}+\mathbf{a}t$ with t = 20 gives $\mathbf{v} = \begin{bmatrix}-8\\6\end{bmatrix} + \begin{bmatrix}0.4\\-0.25\end{bmatrix}20 = \begin{bmatrix}0\\1\end{bmatrix}$ (ms^{-1}).

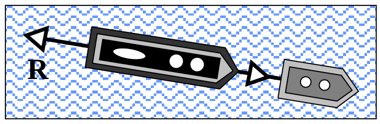
So, after 20 seconds, the skier is travelling at 1 ms^{-1} due north.

Ship

Take the unit vectors $\mathbf{i}$ and $\mathbf{j}$ to be directed east and north respectively.

A ship is pulled by a tug at a constant velocity $\mathbf{u}$ ms^{-1} where $\mathbf{u} = \begin{bmatrix} 2.5 \\ -1 \end{bmatrix}$

against a resistance to motion $\mathbf{R}$ newtons, where $\mathbf{R} = \begin{bmatrix} -5600 \\ 1200 \end{bmatrix}$.



a What is the force, $\mathbf{F}$, provided by the tug?

b The ship's initial position vector relative to a lighthouse, O, is $\mathbf{r}$ metres where $\mathbf{r} = \begin{bmatrix} 300 \\ 500 \end{bmatrix}$.

i Find an expression for the position vector of the boat at time t.

ii The ship is aiming for a buoy which has position vector $\begin{bmatrix} x \\ 100 \end{bmatrix}$ relative to the lighthouse, O. Assuming the ship reaches the buoy, find x.

Ship solution

a As the ship is moving with constant velocity, there is no acceleration, so there must be no resultant force.

This means that the force provided by the tug must be equal in magnitude to the resistance R, but acting in the opposite direction.

So $\quad \mathbf{F} = -\mathbf{R} = \begin{bmatrix} 5600 \\ -1200 \end{bmatrix}$.

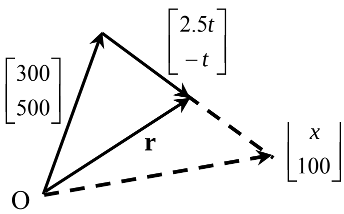
b i Using $\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$ with $\mathbf{a} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

gives $\mathbf{s} = \begin{bmatrix} 2.5 \\ -1 \end{bmatrix} t + \frac{1}{2}\begin{bmatrix} 0 \\ 0 \end{bmatrix} t^2 = \begin{bmatrix} 2.5t \\ -t \end{bmatrix}$.

To find the position vector of the ship at time t, add the displacement $\mathbf{s}$ to the initial position vector.

So, at time t, the position vector of the ship is given by

$$\mathbf{r} = \begin{bmatrix} 300 \\ 500 \end{bmatrix} + \begin{bmatrix} 2.5t \\ -t \end{bmatrix} = \begin{bmatrix} 300 + 2.5t \\ 500 - t \end{bmatrix}$$



ii Using the $\mathbf{j}$ component of the position vector:
The ship reaches the buoy when $500 - t = 100$, giving $t = 400$.

Then using the $\mathbf{i}$ component: $x = 300 + 2.5t = 300 + 2.5 \times 400 = 1300$.

Try these

1 The unit vectors **i** and **j** are directed east and north respectively.
Find the magnitude and direction of each of the following vectors.

a the displacement $\begin{bmatrix} 120 \\ 160 \end{bmatrix}$ m

b the velocity $\begin{bmatrix} -12 \\ 8 \end{bmatrix}$ ms^{-1}

c the acceleration $\begin{bmatrix} 1.5 \\ -2.4 \end{bmatrix}$ ms^{-2}

d the force $\begin{bmatrix} -70 \\ -96 \end{bmatrix}$ N

2 Sally is cycling across the desert.

Her initial velocity **u** ms^{-1} is given by **u** = $\begin{bmatrix} 8 \\ -4 \end{bmatrix}$,

where the unit vectors **i** and **j** are directed east and north respectively.

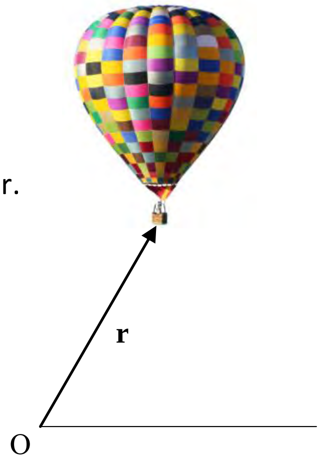
Sally cycles with constant acceleration **a** ms^{-2} where **a** = $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

a Find an expression for Sally's velocity at time *t*.
b i Find her velocity when *t* = 4.
b ii State the direction in which she is travelling at that time.
c The mass of Sally and her cycle is 120 kg.

Find the magnitude of the resultant force that acts during the acceleration.

3 In this question **i** is a horizontal unit vector and **j** is a vertical unit vector.

Relative to point O on horizontal ground, the initial position vector of a hot air balloon is r metres where **r** = $\begin{bmatrix} 50 \\ 120 \end{bmatrix}$.



The balloon is travelling with constant velocity **u** ms^{-1} where **u** = $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

Find …
a the position vector of the balloon at time *t*
b the value of *t* when the balloon reaches the ground.
c the distance between O and the point where the balloon reaches the ground.

4 The unit vectors **i** and **j** are directed east and north respectively.

A motor boat passes through point O with velocity **u** ms^{-1} where **u** = $\begin{bmatrix} -20 \\ 10 \end{bmatrix}$.

The motor boat's acceleration is **a** ms^{-2} where **a** = $\begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix}$.

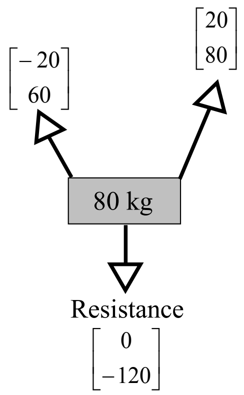
a Find an expression for the position vector of the motor boat, relative to O, at time t.

b Find the position vector of the motor boat, relative to O, when $t = 10$.

c The position vector of a buoy relative to O is $\begin{bmatrix} 420 \\ 450 \end{bmatrix}$.

Find the distance between the motor boat and the buoy when $t = 10$.

5 The unit vectors **i** and **j** are directed east and north respectively.



Two men pull a trolley of mass 80 kg from rest across horizontal ground. The diagram shows the forces (in newtons) applied by the men and the resistance against the motion.

a Find …
- **i** the resultant force
- **ii** the acceleration
- **iii** the speed of the trolley after 6 seconds
- **iv** the direction in which the trolley moves.

b The men stop pulling after 6 seconds.
- **i** How long does it take the trolley to come to rest?
- **ii** Find the total distance it has travelled.

6 The unit vectors **i** and **j** are directed east and north respectively.
The team is rowing a boat.

Initially, its velocity **u** ms^{-1} is given by **u** = $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$.

The boat is moving on a calm lake with constant acceleration **a** ms^{-2} where **a** = $\begin{bmatrix} 0.2 \\ -0.5 \end{bmatrix}$.

a Find an expression for the velocity of the boat at time t.

b Find the velocity when $t = 4$ and state the direction in which the boat is travelling.

c The mass of the team and the boat is 540 kg. Find:

i the force acting on the team and the boat, giving your answer in vector form.

ii the magnitude of this force.

7 The unit vectors $\mathbf{i}$ and $\mathbf{j}$ are horizontal and vertical respectively.

A footballer kicks a ball from a point O on horizontal ground giving it an initial velocity $\mathbf{u}$ ms^{-1} where $\mathbf{u} = \begin{bmatrix} 36 \\ 12 \end{bmatrix}$.

The football's acceleration is $\mathbf{a}$ ms^{-2} where $\mathbf{a} = \begin{bmatrix} 0 \\ -9.8 \end{bmatrix}$.

Find …

a an expression for the velocity of the football at time t

b an expression for the displacement of the ball from O at time t

c the value of t when the ball reaches its greatest height

d the greatest height reached by the ball

e the horizontal distance from O where the ball lands.

8 The unit vectors $\mathbf{i}$ and $\mathbf{j}$ are directed east and north respectively.

A skater is travelling across a frozen pond with constant velocity $\mathbf{u}$ ms^{-1} where $\mathbf{u} = \begin{bmatrix} 2.5 \\ 2 \end{bmatrix}$.

Relative to a point O at the side of the pond, the skater's initial position vector is $\mathbf{r}$ metres where $\mathbf{r} = \begin{bmatrix} 30 \\ 50 \end{bmatrix}$.

a i Find an expression for the position vector of the skater relative to O at time t.

ii Find the position vector of the skater relative to O after 1 minute i.e. when $t = 60$.

b The skater is aiming for a point on the other side of the pond with position vector $\begin{bmatrix} 500 \\ y \end{bmatrix}$.

Assuming the skater reaches this point, find y.

Extensions

Can you see any similarities between the scenarios in the questions you have answered?

If you drew diagrams to help you to answer any of these questions, can you see any similarities between the diagrams for the different problems?

What general principles can you draw from these similarities?

If you have already studied projectiles, solve Question **7** again using the techniques you have learned for dealing with projectiles.

What do you notice if you compare the two methods?

Reflect on your work

How have you used the fact that **i** and **j** are perpendicular unit vectors?

Can you think of other scenarios that could be tackled using vectors in component form?